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Finite theories and marginal operators on the brane

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Abstract: We show how to use D and NS fivebranes in Type IIB superstring theory to construct large classes of finite $\mathcal{N}=1$ supersymmetric four dimensional field theories. In this construction, the beta functions of the theories are directly related to the bending of branes; in finite theories the branes are not bent, and vice versa. Many of these theories have multiple dimensionless couplings. A group of duality transformations acts on the space of dimensionless couplings; for a large subclass of models, this group always includes an overall $SL(2,\mathbb{Z})$ invariance. In addition, we find even larger classes of theories which, although not finite, also have one or more marginal operators.

Keywords: Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories, Gauge Symmetry, Supersymmetry and Duality.

Contents

Most quantum field theories do not contain adjustable dimensionless couplings. Even conformal field theories are generally isolated points in the space of all possible models. Field theories with exactly marginal operators, including finite theories, are interesting because they contain truly dimensionless continuous parameters. Many examples are known in two, three and four dimensions. Our focus in this paper will be the four-dimensional case.

One reason why dimensionless couplings are worthy of study is that duality transformations can act on them. The classic example is the action of $SL(2,\mathbb{Z})$ on the coupling $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$ in $\mathcal{N}=4$ supersymmetric theories. $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetric gauge theories are known in which a duality group G_D acts on a complex coupling constant [1]–[9]. In the $\mathcal{N}=1$ case, many of the theories with marginal couplings are not finite; instead, these theories have non-trivial beta functions whose zeroes occur when some fields' anomalous dimensions are non-zero [5, 6, 7]. Often finite theories with marginal couplings are dual (using $\mathcal{N}=1$ duality [10]) to non-finite theories with marginal couplings [5, 6, 11]. If one wants to characterize the space of theories and how different duality transformations act on this space — which may be relevant for the recent work relating string theory in ten dimensions to gauge theory in four—then understanding these theories is important.[1]

Recently, constructions of field theories using branes of string theory and M theory have proven to be a powerful tool for studying properties both of the field theories and of the branes; for a recent review, see [13]. In this paper, we show that the Type IIB fivebrane constructions of $\mathcal{N}=1$ supersymmetric field theories can be used to generate large classes of finite models, and even larger classes of models which are not finite but still have marginal operators. In sect. 1 we discuss the criteria for marginal couplings in general and finiteness in particular. Next, in sect. 2, we discuss the brane construction of a large set of field theories. We then identify a class of theories in sect. 3 which satisfy the criteria for containing at least one marginal operator, and consider a number of examples, many of which are finite. We will see both "elliptic" models (generalizations of $\mathcal{N}=4$ supersymmetric $SU(N)$ gauge theories) and "cylindrical" models (generalizations of $\mathcal{N}=2$ supersymmetric $SU(N)$ theories with $2N$ hypermultiplets) which are degenerations of the elliptic models. We discuss the duality properties of these models in sect. 4. As yet we have only a partial understanding of the coupling constants and duality transformations that these theories possess; we discuss some unresolved issues in Appendix A. Proofs of various claims appear in Appendix B.[2]

1. Field theory considerations

In four dimensions all known examples of theories with marginal operators have four or more supersymmetries.[3] Finite theories in $\mathcal{N}=1$ supersymmetry have a long history and proofs have been given that they are finite to all orders. Particularly clear are the proofs of Lucchesi, Piguet and Sibold [18, 19]. However, the approach of [5], which is similar in some ways to [18, 19], has the advantages that it is extremely simple to state, is non-perturbative in form, and generalizes to non-finite theories.

[1]It has also long been argued that finite grand unified theories might be relevant for phenomenology; for a recent discussion see [12]. However, given the existence of gravity as a natural cutoff for field theory, and given the finiteness of string theory, we see no particular compelling argument for finite field theories below the string scale.

[2]After this work was substantially completed, a number of papers on related subjects appeared [14, 15, 16] whose motivation is completely different but whose results are related to our own. The finite models discussed here appeared in the classification of [16]. More closely related work appeared in another recent paper [17].

[3]Proposals for finite non-supersymmetric field theories have recently been made [15, 16], though they likely are finite only when the number of colors is strictly infinite.

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There is a very simple criterion that tests whether an $\mathcal{N}=1$ supersymmetric theory may have a marginal operator [5]. More details and additional references are given in [5, 20]. If the superpotential contains a term $W = h\phi_1 \ldots \phi_n$, then the beta function corresponding to the coupling h is

$$\beta_h \propto A(h) \equiv (n-3) + \frac{1}{2}\sum_{k=1}^{n} \gamma(\phi_k)\,, \tag{1.1}$$

where $\gamma(\phi_k)$ is the anomalous mass dimension of the superfield ϕ_k. For the gauge coupling of a gauge group G, the formula for the beta function is [21, 22, 23]

$$\beta_g \propto A(g) \equiv -\left\{\left[3C_2(G) - \sum_j T(R_j)\right] + \sum_j T(R_j)\gamma(\phi_j)\right\}\,, \tag{1.2}$$

where the sums are over all matter fields, $C_2(G)$ is the quadratic Casimir of the adjoint representation of G, and $T(R_j)$ is the quadratic Casimir of the representation R_j of G in which ϕ_j transforms. The term in square brackets is the one-loop beta function coefficient b_0.

If we have a theory with p Yukawa couplings h_i and q gauge couplings g_j, the anomalous dimensions are unknown real functions of the couplings h_i and g_j. The condition for a conformal fixed point is that all beta functions β_{h_i}, β_{g_j} vanish, putting $p+q$ conditions on $p+q$ couplings. The generic theory has isolated solutions to these conditions. However, because *the beta functions are linear functionals of the anomalous dimensions*, it may happen that the $p+q$ conditions $\beta_{h_i} = 0 = \beta_{g_j}$ are linearly dependent. If only $p+q-r$ of these conditions are linearly independent, then the solutions will form an r complex-dimensional subspace of the $p+q$ complex dimensional space of couplings. These solutions will represent a set of conformal field theories with r adjustable dimensionless couplings. Of course, a given theory may have *no* interacting conformal fixed points anywhere in coupling space.

If the conditions for a fixed point also imply that all the anomalous dimensions vanish, then the theory has no divergences in perturbation theory (except those which appear in composite operators) and is as finite as $\mathcal{N}=4$ supersymmetric gauge theory. In this case the manifold of fixed points necessarily includes the free theory. Note that two necessary conditions for finiteness are that all couplings in the superpotential be dimensionless at the classical level and that all gauge beta functions vanish at one loop.[4]

Lists of finite models may be found in [25]. Many examples of theories with marginal operators, including finite models, are given in [5]. Here we mention a couple of well-

[4]It is proven that if there is a choice of couplings such that all beta functions vanish and all anomalous dimensions vanish at leading order, then the theory is finite to all orders [19]. There is also a proof [24] that if the manifold of fixed points includes the free theory, then the anomalous dimensions vanish at leading order. Thus, when the conditions of [5] for vanishing beta functions permit the possibility that the anomalous dimensions vanish, that possibility is apparently always realized. A profound understanding of why this is true is lacking.

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known cases. Consider $SU(3)$ with nine triplets Q^i and nine antitriplets $\tilde{Q}_j$ [26, 27, 28]. This theory is finite if the superpotential

$$W = h\left(Q^1Q^2Q^3 + Q^4Q^5Q^6 + Q^7Q^8Q^9 + \tilde{Q}_1\tilde{Q}_2\tilde{Q}_3 + \tilde{Q}_4\tilde{Q}_5\tilde{Q}_6 + \tilde{Q}_7\tilde{Q}_8\tilde{Q}_9\right) \qquad (1.3)$$

is added. The flavor symmetries among the matter fields ensure they all have the same anomalous dimension $\gamma(g,h)$. (Note that if the flavor symmetries were slightly broken by the superpotential, they would be restored in the infrared [5].) Since $\beta_g \propto \beta_h \propto \gamma(g,h)$, the theory is conformal and has no perturbative divergences when $\gamma(g,h) = 0$ — a single condition on two couplings, leading to a one-complex-dimensional space of solutions. It is easy to show that γ has a zero in perturbation theory.

Other similar examples include E_6 with twelve fields Q^i in the **27** representation with superpotential

$$W = h_1 \sum_{r=1}^{12} (Q^r)^3 + h_2 \sum_{r=1}^{4} Q^r Q^{r+1} Q^{r+2} \qquad (1.4)$$

and $SU(N)$ theories with $\mathcal{N} = 4$ matter content (three adjoint chiral multiplets Φ_1, Φ_2, Φ_3) with superpotential

$$W = (h_1\ f^{abc} + h_2\ d^{abc})\Phi_1^a\Phi_2^b\Phi_3^c + h_3 d^{abc} \sum_{i=1}^{3} \Phi_i^a\Phi_i^b\Phi_i^c\,. \qquad (1.5)$$

Both of these have multiple marginal operators. Note also that the E_6 model is chiral.

An example of a model that is not finite but has a marginal operator is $SU(4)$ with eight flavors and a quartic superpotential [5]

$$W = h(Q^1Q^2Q^3Q^4 + Q^5Q^6Q^7Q^8 + \tilde{Q}_1\tilde{Q}_2\tilde{Q}_3\tilde{Q}_4 + \tilde{Q}_5\tilde{Q}_6\tilde{Q}_7\tilde{Q}_8)\,. \qquad (1.6)$$

This model is a continuous deformation of the theory with vanishing superpotential, which is believed to have a fixed point at some value of the gauge coupling [10].

There is also a link between marginal operators and the duality of a wide class of $\mathcal{N} = 1$ models. In [5], a close connection was noted between the duality of finite $\mathcal{N} = 2$ $SU(N)$ theories [1, 2, 3] and duality of non-finite $\mathcal{N} = 1$ $SU(N)$ models with marginal operators. This connection was further explored in [29, 30, 31, 32].

We now turn to the construction of large classes of these models, both finite and otherwise, using fivebranes in Type IIB string theory. It turns out that the rules for finiteness and for marginal operators in these theories translate into simple geometrical and algebraic statements.

2. Brane construction

2.1. Field theories from fivebranes

The finite $\mathcal{N} = 1$ models we will be considering are constructed in a brane setup, in the spirit of [33], which was described in detail in [34]. The description here will be short and further details are contained in [34].

We are working in Type IIB superstring theory with the following set of branes.

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- NS branes along 012345 directions

- NS$'$ branes along 012367 directions

- D5 branes along 012346 direction.

The D5 branes will be finite in two of the directions, 4 and 6; their low-energy effective world volume theory is 3+1 dimensional. The presence of all branes breaks supersymmetry to 1/8 of the original supersymmetry, and thus we are dealing with $\mathcal{N} = 1$ supersymmetry (4 supercharges) in four dimensions.

In most of the applications in this paper we will be working on a torus in the 46 directions. We consider the 46 plane $\mathbb{R}^2$ divided into a grid by an infinite number of NS and NS$'$ branes. In each box of the grid we will place a number of D5 branes. If the assemblage has translational symmetries, which form a lattice group Λ generated by two shift vectors v_1 and v_2, then we can construct a torus $T^2 = \mathbb{R}^2/\Lambda$ which consists of the unit cell of the lattice Λ. The 4 and 6 directions will become circles with radii R_4 and R_6.

A generic configuration then consists of a grid of k NS branes and k' NS$'$ branes, dividing the 46 torus into a set of kk' boxes. In each box, we can place an arbitrary number of D5 branes. Let $n_{i,j}$ denote the number of D5 branes in the box i,j, $i = 1,\ldots,k, j = 1,\ldots,k'$. In the following, indices will denote variables in a periodic fashion: an index i is to be understood modulo k and an index j is to be understood modulo k'. Thus a model's gauge and matter content is specified by the numbers k and k' and the set of numbers $\{n_{i,j}\}$.

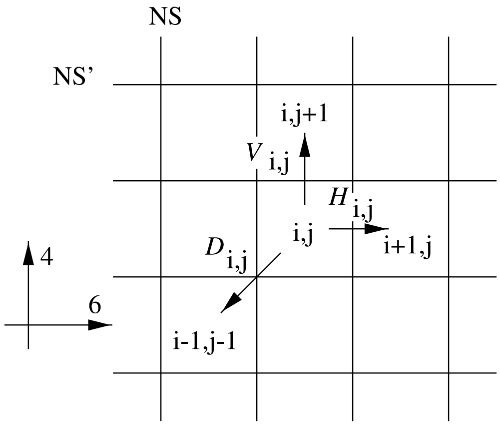

Figure 1: The NS and NS$'$ branes form a grid; in each box (i,j) of the grid lie $n_{i,j}$ D5 branes. The arrows denote the chiral multiplets $H_{i,j}, V_{i,j}, D_{i,j}$ which are in the fundamental of the group $SU(n_{i,j})$ and in the antifundamental of an adjacent group. The arrows at the far left indicate the x^4 and x^6 coordinates.

The gauge group is $\prod_{i,j} SU(n_{i,j})$. (Classically the gauge group also includes one $U(1)$ factor for each box, though these are not present quantum mechanically; see the discussion in subsect. 2.4.) The matter content of the model consists of three types of $\mathcal{N} = 1$ chiral representations. They will be denoted as $H_{i,j}$, $V_{i,j}$ and $D_{i,j}$, corresponding to the horizontal, vertical and diagonal multiplets which arise in the brane system (see the details in [34]). $H_{i,j}$ transforms in the $(\Box,\bar{\Box})$ of $SU(n_{i,j}) \times SU(n_{i+1,j})$, $V_{i,j}$ transforms in the $(\Box,\bar{\Box})$ of $SU(n_{i,j}) \times SU(n_{i,j+1})$ and $D_{i,j}$ transforms in the $(\Box,\bar{\Box})$ of $SU(n_{i,j}) \times SU(n_{i-1,j-1})$. Figure 1 shows the conventions for denoting the multiplets.

The superpotential in these models is calculated using the rules described in [34]. It is given by

$$W=\sum_{i,j}H_{i,j}V_{i+1,j}D_{i+1,j+1}-\sum_{i,j}H_{i,j+1}V_{i,j}D_{i+1,j+1}\,. \tag{2.1}$$

The first term corresponds to lower triangles of arrows and the second term corresponds to upper triangles of arrows in the notation of [34], as shown in fig. 2. Note the relative minus sign between the two terms. By symmetry, the signs are obviously independent of i and j; the relative sign is determined by looking at the special case $k=k'=1$, which will be discussed in more detail below. From a general field theory point of view, there may be coefficients $h^+_{i,j}$ and $h^-_{i,j}$ for each Yukawa term. It is convenient for now to set these coefficients to one by a redefinition of the fields.

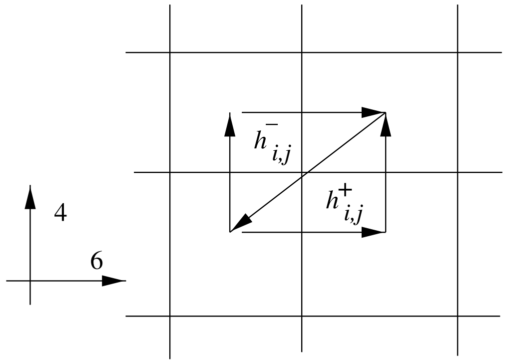

Figure 2: The two superpotential interactions at each corner, with couplings $h^+_{i,j}$ and $h^-_{i,j}$, are represented by an oriented triangle of arrows.

Our understanding of the gauge couplings, especially the theta angles, is incomplete. However, some contributions to these couplings can be characterized. In the simplest situations, the gauge couplings of the various gauge groups are given by the positions of the NS branes in the x^6 direction and the position of the NS$'$ branes in the x^4 direction. There are k positions x_6^i and k' positions x_4^j. Correspondingly, the x_6 direction is divided into k intervals with lengths $a_i=x_6^i-x_6^{i-1}$, such that $\sum_i a_i=R_6$. The x_4 direction is divided into k' intervals of length $b_j=x_4^j-x_4^{j-1}$, such that $\sum_j b_j=R_4$. The gauge coupling $g_{i,j}$ for the group $SU(n_{i,j})$ is given by

$$\frac{1}{g_{i,j}^2}=\frac{a_ib_j}{g_sl_s^2}\,. \tag{2.2}$$

The kk' gauge couplings are not all independent. They are given by $k+k'-1$ parameters corresponding to the positions of the NS and NS$'$ branes. Two positions can be set to zero by the choice of origin in the 46 directions, but the area of the 46 torus gives one more parameter. The couplings do not depend on the ratio between the two radii of the torus. As we will see later, the field theories often have more than $k+k'-1$ dimensionless parameters, indicating that we have not identified all of the contributions to these couplings.

The theta angles of the gauge theories receive various contributions. Let A_i be the gauge field on the world volume of the i^{th} NS brane and A'_j be the gauge field on the world volume of the j^{th} NS$'$ brane. Since the dimensions 4 and 6 are compact, there can be non-zero Wilson lines of A_i along 4, and of A'_j along 6. Let $R_{i,j}$ denote the area in the 46 direction which is bounded by the pair of NS branes and NS$'$ branes. The theta angle for the i,j group depends on the line integral of the different gauge fields along the boundary of $R_{i,j}$. Schematically,

$$\theta_{i,j} = \int_{R_{i,j}} B + \int_{a_i} (A'_{j-1} - A'_j) + \int_{b_j} (A_i - A_{i-1}) \,, \tag{2.3}$$

where B is the RR two form of Type IIB superstring theory. The contributions from the gauge fields are required for the invariance of $\theta_{i,j}$ under gauge transformations of B. Were this the entire story we would again have $k + k' - 1$ parameters. However, invariance under gauge transformations of the one-forms require that additional terms be added to this expression involving axion-like fields living at the intersections of the NS and NS$'$ branes. Some additional discussion of this issue is presented in Appendix A.

2.2. An example

Let us specify this in a concrete case, with two NS branes of each kind, $k = k' = 2$, and an equal number N of D5 branes in each box. The torus is the grid identified under shifts by two boxes vertically and by two boxes horizontally. The unit cell's four boxes, eight Yukawa interactions and twelve matter fields are shown in fig. 3. This gives an $\mathcal{N} = 1$ gauge theory with gauge group $SU(N) \times SU(N) \times SU(N) \times SU(N)$, with vectorlike matter content

$$\begin{aligned}
H_{1,1} &= \square, \overline{\square}, 1, 1 = \overline{H_{2,1}} \\
V_{1,1} &= \square, 1, \overline{\square}, 1 = \overline{V_{1,2}} \\
D_{1,1} &= \square, 1, 1, \overline{\square} = \overline{D_{2,2}} \\
H_{1,2} &= 1, 1, \square, \overline{\square} = \overline{H_{2,2}} \\
V_{2,1} &= 1, \square, 1, \overline{\square} = \overline{V_{2,2}} \\
D_{2,1} &= 1, \square, \overline{\square}, 1 = \overline{D_{1,2}}
\end{aligned} \tag{2.4}$$

The superpotential is as in eq. (2.1). In the simplest situation, the four gauge couplings are specified by the areas of the boxes.

2.3. Bending and the beta function

It is important to note that there is a geometrical picture for the beta function in this class of models. This is a well-known property of theories with eight supercharges, which we review. Consider the constructions of field theories using D branes on a one-dimensional interval (as opposed to the two-dimensional intervals used in this paper), as in [33] and its various generalizations; see [13] for a review. A field theory in d dimensions with 8 supercharges is realized on the world volume of a Dirichlet d brane which is bounded by two NS branes in one direction. When the Dd brane ends on the NS brane there is a local bending of the NS brane which goes like r^{d-4}, r denoting the radial coordinate in the subspace of the NS world volume transverse to the Dd brane. We can measure the asymptotic bending by going to large r, along the world volume of the NS brane. On the other hand, the transverse distance between the two NS branes corresponds to the gauge coupling of the system. The asymptotic bending of the NS branes controls the evolution of the gauge coupling with scale. Thus, when there is bending which moves the two NS branes towards each other, the coupling grows at high energies and we are dealing with a IR free theory. When the bending is outwards, the coupling becomes weak in the UV and we are dealing with an asymptotically free theory. Let us check what happens for various dimensions. From the local bending behavior, we see that for $d > 4$ there is an asymptotic bending of the NS brane. For $d < 4$ there is no asymptotic bending. For $d = 4$, the bending is logarithmic and causes the NS brane to bend away from the D brane [4].

From a field theory point of view, it is known that all gauge theories in dimensions $d < 4$ are asymptotically free, and that all field theories with finite gauge couplings are infrared free for $d > 4$; thus we get perfect agreement with the brane picture. For $d = 4$, the situation is more interesting, because a given NS brane may bend to the left (right) at long distances if the number of D4 branes intersecting it from the right is greater (less) than the number of D4 branes intersecting it from its left — that is, if its linking number [33] is positive (negative). A given set of parallel D branes between two NS branes will represent an asymptotically free gauge group if the distance between the two NS branes grows with r, an infrared free theory if it shrinks, and a finite theory if the two NS branes are parallel at large r. Thus, the relative distances between NS branes far from the D4 branes reflects the beta functions of the gauge groups lying between them, giving an intuitive correspondence between bending and the beta function. In particular, when all of the NS branes are unbent, parallel to each other, and perpendicular to the D4 branes — which requires that the number of D4 branes be the same in all the intervals between NS branes — then all the beta functions are vanishing and the corresponding theory is finite. Such models were discussed in [4].

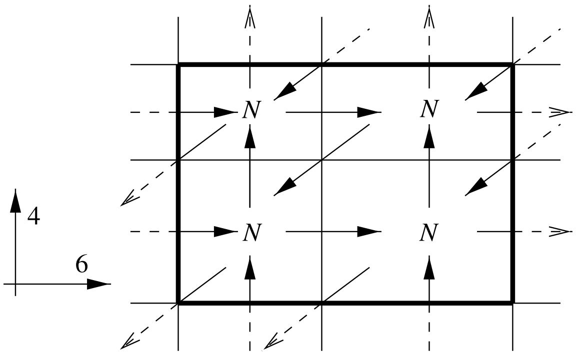

Figure 3: Construction of a model on a torus, expressed as a unit cell of a two-dimensional lattice. The unit cell of four boxes is highlighted. Note the twelve matter multiplets, indicated by the solid parts of the arrows; the dotted parts of the arrows correspond to their images under shifts of the unit cell by lattice vectors.

Unfortunately, when the branes are rotated, as in [35], breaking the supersymmetry down to $\mathcal{N} = 1$ in four dimensions, the bending of the NS branes does not represent the beta function any more. Instead, the bending of the branes corresponds to the R charges of the various fields; see for example [13].

However, in our construction, where we represent $\mathcal{N} = 1$ four-dimensional models using D5 branes on a two-dimensional interval, the bending and beta functions are again related. As described in [34], all the configurations drawn in this paper are given for zero string coupling. When the string coupling is non-zero the branes start to bend. The problem of bending is not fully solved, but we can make some simple statements.

First, we can discuss simple configurations in which bending will not be present. In analogy to the case just discussed, the NS branes will not bend at all if the number of D5 branes to their right and left are everywhere equal. A similar statement applies to the NS′ branes. Clearly, for there to be no bending anywhere, the number of D5 branes in every box should be the same. As we will see below, this is indeed the condition for a finite $\mathcal{N}=1$ gauge theory! Unlike the case of eight supersymmetries, however, this is not a straightforward remark. With eight supersymmetries, the finiteness of the theory is determined by the one-loop beta functions only. With four, one must consider the interactions in the superpotential more carefully and ensure that the conditions discussed in sect. 1 are satisfied. It turns out that the brane configuration automatically adds the correct superpotential and satisfies these conditions, as is proven in Appendix B and discussed in the next section.

Second, it is easy to show that, far from the D5 branes, the NS and NS′ branes tend to bend toward (away from) each other for infrared (asymptotically) free theories. For example, consider the case where $n_{2,2}=N$ and $n_{1,1}=n_{1,2}=n_{2,1}=n_{2,3}=n_{3,3}=n_{3,2}=p$. The $(2,2)$ box then corresponds to an $\mathcal{N}=1$ $SU(N)$ gauge theory with $3p$ flavors, with one-loop beta function coefficient $b_0=3(N-p)$. Meanwhile, if $p>N$ ($p<N$), then at each edge of the $2,2$ box the number of D5 branes outside the box is greater (less than) the number inside, so the NS and NS′ branes bordering the $2,2$ box will bend inward (outward). Thus the bending of the branes is sensitive to the beta function. However, in situations which are less symmetric, the bending of the branes is not well understood.

To conclude, we propose a very simple criterion for a finite theory. Given a brane configuration which has no bending, the corresponding field theory which is read off from the brane configuration by using the rules of [34] is a finite theory. This criterion is a simple generalization of the concept for theories with 8 supercharges, and gives us a large class of finite $N=1$ theories with almost no effort. For the cases where NS and NS′ branes do bend, we do not yet fully understand the situation, but clearly the behavior of the branes is related to the $\mathcal{N}=1$ beta functions.

2.4. $U(1)$ factors, Fayet-Iliopoulos terms, and flat directions

We now comment on Fayet-Iliopoulos (FI) parameters and the possible "frozen" $U(1)$ fields. The gauge groups for zero string coupling contain $U(1)$ factors. It is argued in [4] that these $U(1)$ fields are frozen from a four dimensional point of view once the string coupling is turned on. We expect a similar situation here. However, for the classical theory, with vanishing string coupling, the $U(1)$ fields are present, and with them, FI parameters.

First we need an expression for the FI parameters. This is determined as follows. For $k=1$ or 0 and any k', the system reduces to a $N=2$ system and we can refer to previous results which give the FI coupling for the j-th $U(1)$ $r_j = x_5^j - x_5^{j-1}$. Here x_5^j is the position of the NS$'$ brane in the 5 direction. Equivalently, for $k' = 1$ or 0 and any k, we have again an $N = 2$ system and the FI term for the i-th $U(1)$ is $r_i = x_7^i - x_7^{i-1}$, with x_7^i the position of the i-th NS brane. An expression which is consistent with these two boundary conditions and assumes a linear relation gives for the FI coupling of the i, j-th gauge group $r_{i,j} = x_5^j - x_5^{j-1} + x_7^i - x_7^{i-1}$.

We can further check this proposal for the FI couplings by considering the simple breaking of gauge symmetries generated by sending one NS brane to far infinity in the 7 direction and reproducing the same formulas for couplings. The associated FI terms lead to the correct breaking pattern in the classical field theory. In the quantum theory, the $U(1)$ fields will all be frozen, so instead the position of the NS brane gets the interpretation of a baryonic expectation value. However, the pattern of symmetry breaking is the same.

3. Marginal operators

In this section we first explain the conditions for field theories to have dimensionless couplings and associated marginal operators, and then give a number of examples.

Although we will be interested in models with $k \times k'$ boxes on a torus, it is easiest to begin our investigations by considering an infinite grid of NS and NS$'$ branes, with $n_{i,j} \geq 2$ D5 branes filling the square (i, j). The field theory defined by this infinite arrangement has at least one marginal coupling if, for every 2×2 block of squares, the following condition holds:

$$n_{i,j} + n_{i+1,j+1} = n_{i+1,j} + n_{i,j+1} \, . \tag{3.1}$$

The proof of the above claim is given in Appendix B.

Note that these (generically) chiral field theories are always anomaly free. For any square (i, j), the number of fields in the fundamental representation is $n_{i,j+1} + n_{i+1,j} + n_{i-1,j-1}$, while the number in the antifundamental representation is $n_{i,j-1} + n_{i-1,j} + n_{i+1,j+1}$. The conditions (3.1) above ensure these are equal. Furthermore, the condition ensures that the difference between the number of D5 branes on the two sides of an NS or NS$'$ brane remains a constant as one moves along the brane. This ensures that the amount of bending which the NS or NS$'$ brane undergoes remains constant as well.[5]

If we want a theory with no divergences, however, we must require as an additional condition that all of the one-loop beta functions vanish. This demands that

$$n_{i,j+1} + n_{i+1,j} + n_{i-1,j-1} = 3n_{i,j} \tag{3.2}$$

for every i, j, whose only solution is $n_{i,j} = N$ for all N. This ensures that in finite models the NS and NS$'$ branes are unbent, as noted earlier in subsect. 2.3.

[5] The connection between anomalies and bending was pointed out in [17].

In addition, the finite models have still more dimensionless couplings. As shown in Appendix B, there will be one for each row of boxes, one for each column of boxes, and one for each diagonal line running from lower-left to upper-right (that is, along the D fields) passing through the boxes.

As described in sect. 2, the models we want to study lie not on an infinite plane but on a finite torus. If we take a model on the $x_4 - x_6$ plane $\mathbb{R}^2$ whose grid of boxes and choices of $\{n_{i,j}\}$ are left invariant by a discrete lattice group Λ of translations, we may consider the theory on the torus formed by the quotient of the plane by Λ. The resulting theories all have a marginal coupling associated with the overall Kähler parameter ρ of the torus. In addition, there may be additional marginal couplings in special cases. In the particular case where all boxes have N D5 branes, we find a *finite* theory, which in addition to the marginal coupling associated with ρ has marginal couplings for every independent row, column, and lower-left/upper-right diagonal line passing through the boxes.

We will now give some examples of such models.

3.1. $\mathcal{N} = 4$ $SU(N)$ gauge theory

First consider $k = k' = 1$, as shown in fig. 4. This choice gives a single gauge group $SU(N)$ with $\mathcal{N} = 4$ supersymmetry. The three fields H, V and D are charged under the same gauge group because of the periodicity of the torus in the x_4, x_6 directions. Being bi-fundamentals under the same group, they are adjoint fields Φ_i, $i = 1, 2, 3$, giving the matter content of a $\mathcal{N} = 4$ theory. (Again, classically these fields are adjoints of $U(N)$, and so contain a gauge singlet for each of H, V and D; however these fields simply become the x_7 (x_5) position of the NS (NS′) brane in the quantum theory.) Furthermore, the superpotential (2.1) in this case reads $W = H[V, D]$, reproducing the appropriate $\mathcal{N} = 4$ interaction. Note the importance of the relative minus sign in (2.1) between upper and lower triangle contributions.

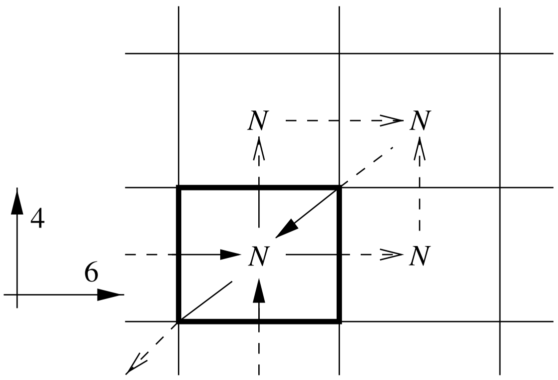

Figure 4: The $\mathcal{N} = 4$ gauge theory; the torus consists of the one highlighted box, of which the other boxes are images. The three arrows correspond to the three chiral multiplets in the adjoint representation.

The gauge coupling, equation (2.2), takes the form

$$\frac{1}{g^2} = \frac{R_4 R_6}{g_s l_s^2} \tag{3.3}$$

and the theta angle, eq. (2.3), is simply given as the integral of the RR B field on the torus

$$\theta = \int_{46} B \,. \tag{3.4}$$

The complex gauge coupling constant τ is given by the combination of these parameters, which corresponds to the Kähler parameter of the torus. In the following we denote it by ρ to avoid confusion with the traditional name for the complex structure parameter of a torus.

Note that removing the NS and/or the NS′ brane from the torus does not change the group; it merely corresponds to changing the expectation value of the neutral scalars as discussed in subsect. 2.4. Without the NS and NS′ brane, the D5 brane wrapped on the torus is by T duality equivalent to a D3 brane, and the Kähler parameter of the torus becomes the IIB string coupling constant. This is consistent with the construction of $\mathcal{N}=4$ gauge theories from D3 branes, and provides a check on our construction.

3.2. Elliptic $\mathcal{N}=2$ models

Consider a model on a torus identified under vertical shifts by one box and under horizontal shifts by k boxes. Each of the k boxes contains N D5 branes, as shown in fig. 5. (The construction has k NS branes and one NS′ brane; however, as in the previous case, the one NS′ brane may be removed without changing the theory.) This is an $\mathcal{N}=2$ supersymmetric $SU(N)\times SU(N)\times\cdots\times SU(N)$ gauge theory, with hypermultiplets in bifundamental representations. The $\mathcal{N}=1$ chiral multiplet in the $\mathcal{N}=2$ $SU(N)_i$ vector multiplet is the field $V_{i,1}$, while the hypermultiplets under $SU(N)_i\times SU(N)_{i+1}$ are the fields $(H_{i,1}, D_{i+1,1})$. The $\mathcal{N}=1$ superpotential for this model also reproduces the correct $\mathcal{N}=2$ interactions. The model has k dimensionless

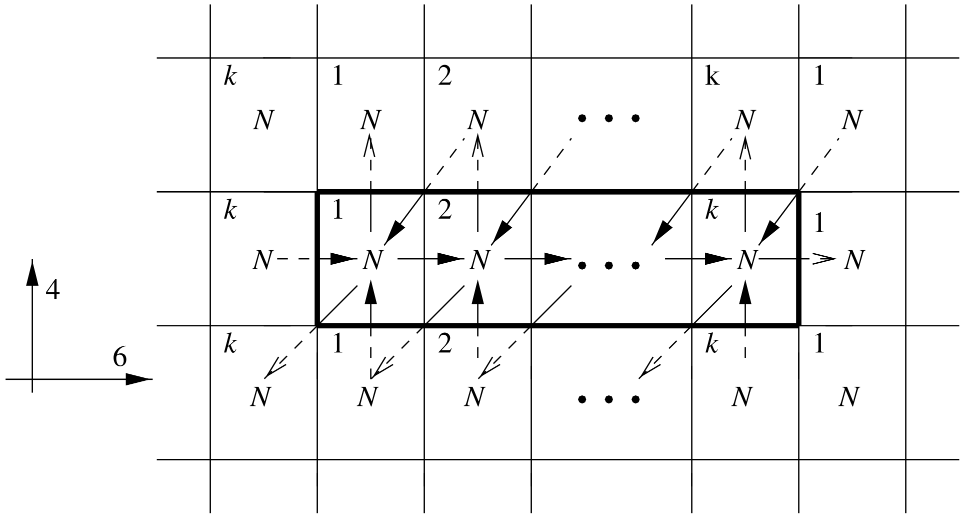

Figure 5: Construction of the $\mathcal{N}=2$ elliptic models. The theory has gauge group $SU(N)_1\times SU(N)_2\times\cdots\times SU(N)_k$; each box corresponds to one of these $SU(N)$ factors, as indicated by the indices in the upper left corners.

couplings, corresponding to the overall size of the torus and to $k-1$ motions in x_6 of the NS branes (these couplings are complexified by the parameters from the Wilson lines of the NS brane world-volume gauge fields, see Appendix A.) These models are the same as the elliptic models of [4], to which they are related by T duality in the x_4 direction.

These models have flat directions (classically they are Fayet-Iliopolous parameters of the $U(1)$ groups which are absent quantum mechanically) which break the k-box model to the $k-1$ box model; these motions correspond to moving one of the NS branes off of the D5 branes. If all but one or zero of the NS branes is removed, the resulting 1-box model is the $\mathcal{N}=4$ gauge theory considered above.

3.3. Elliptic $\mathcal{N}=1$ models

A number of elliptic finite $\mathcal{N}=1$ supersymmetric models may be constructed in this way, by considering a torus of $k \times k'$ boxes with each box filled with N D5 branes. In this section we consider a few examples.

Consider first the $k=k'=2$ model of fig. 3. This theory is vectorlike: there is a bifundamental and its complex conjugate connecting every pair of groups, for a total of twelve fields. It is easy to check this model is finite, and that it has a total of four dimensionless couplings: one for the overall torus, one for the separation of the NS branes, one for the separation of the NS$'$ branes, and one for the diagonal.

Next consider a $k=k'=3$ model, as in fig. 6, where the grid of boxes is identified under shifts by three boxes vertically and by three boxes horizontally. This model has eighteen fields in chiral representations; there is a bifundamental *or* a biantifundamental connecting each pair of groups except those whose boxes lie in an upper-left/lower-right

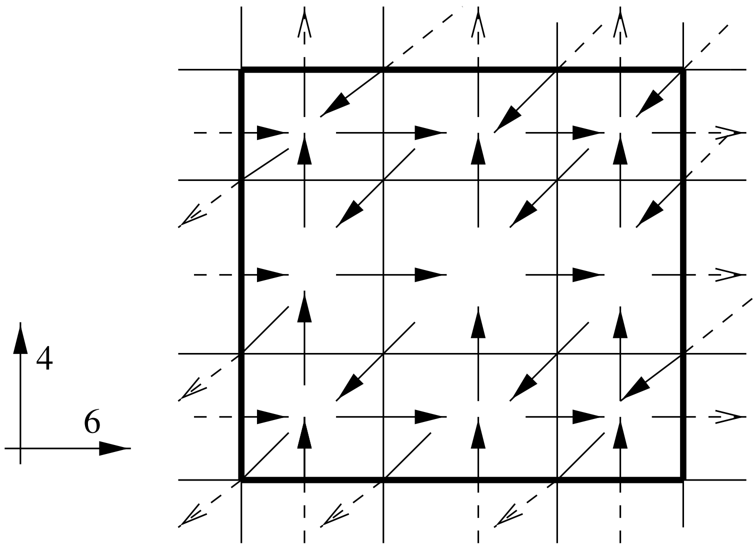

Figure 6: Construction of the 3×3 $[SU(N)]^9$ chiral model.

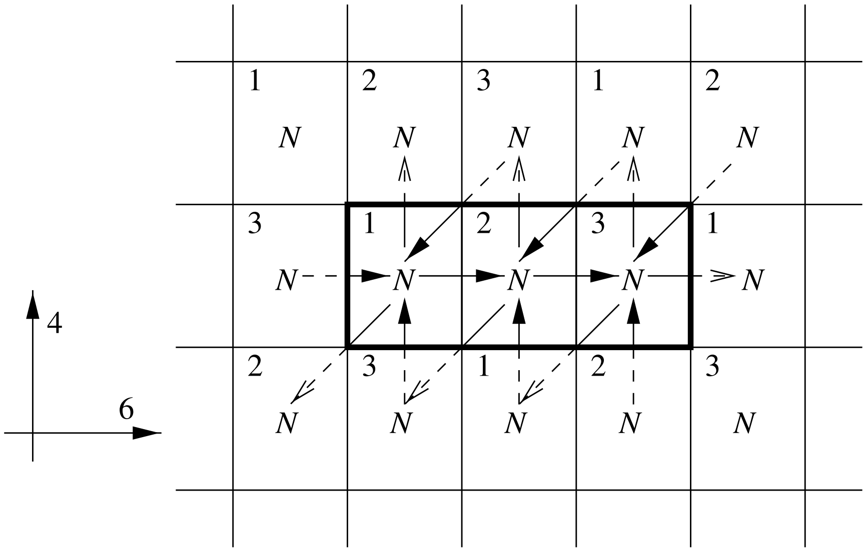

Figure 7: Construction of the 1×3 twisted model. The theory is $SU(N)_1 \times SU(N)_2 \times SU(N)_3$; each box corresponds to one of these $SU(N)$ factors, as indicated by the indices in the upper left corners.

orientation. There are seven dimensionless couplings: two from rows, two from columns, two from diagonals, and one from ρ.

Each of these models has flat directions (or Fayet-Iliopolous parameters) along which it flows to a model with fewer boxes by removal of NS or NS′ branes. For example, one may have the transitions $3 \times 3 \to 3 \times 2 \to 2 \times 2 \to 2 \times 1 \to 1 \times 1$ in which the chiral $\mathcal{N} = 1$ model flows to a vectorlike $\mathcal{N} = 1$ model, which then flows to an $\mathcal{N} = 2$ model and from there to an $\mathcal{N} = 4$ gauge theory.

Next consider a 1×3 model, as in fig. 7, where the torus is obtained by quotienting an infinite grid by the translation group generated by a horizontal translation (of three boxes) and a *diagonal* translation (of one box up and one box to the left). Equivalently, this is the previous 3×3 model modded out by a freely acting Z_3 transformation. The spectrum contains three copies of chiral fields transforming as follows under $SU(N)_1 \times SU(N)_2 \times SU(N)_3$

$$Q_{1a} = (1, \Box, \overline{\Box})\,; \qquad Q_{2a} = (\overline{\Box}, 1, \Box)\,; \qquad Q_{3a} = (\Box, \overline{\Box}, 1)\,, \tag{3.5}$$

with $a = H, V, D$ for horizontal, vertical and diagonal fields. The superpotential can be recast as the simple expression

$$W = \epsilon^{ijk} Q_{iH} Q_{jV} Q_{kD}\,. \tag{3.6}$$

Here there is only one dimensionless coupling; there is only one NS brane and one NS′ brane, and so only the overall size of the torus plays a role. The three gauge couplings are fixed to be equal. (Note that, while this is strictly true of the brane construction for geometrical reasons, the field theory is more general, and only need have equal couplings when the condition of finiteness is imposed.) This model has recently been discussed in [14, 15], where it appeared in the worldvolume of D branes at $\mathbb{C}^3/Z_3$ singularities. It would be interesting to clarify the relation between both approaches. Clearly, many generalizations of this model are possible.

3.4. Cylindrical $\mathcal{N}=2$ and $\mathcal{N}=1$ models

If we allow the torus to degenerate to a cylinder, by allowing the area of the torus to become infinite while keeping R_4 or R_6 finite, we find a new class of models as a limit of our elliptic models.

For illustration we begin with the $\mathcal{N}=2$ case, which is discussed in [4]. If $k=2, k'=1$ in fig. 5, we have an $\mathcal{N}=2$ gauge theory with $SU(N)_1 \times SU(N)_2$ as gauge group. We may imagine letting the x^6 length of the second box grow to infinity while that of the first box remains finite; thus, with the NS branes held at a fixed distance L_6, and with R_4 held constant, R_6 goes to infinity. We are left with a cylinder of radius R_4 with one finite box and two infinite boxes (both corresponding to $SU(N)_2$ on either side). $SU(N)_2$ becomes free in this limit and becomes a global symmetry of $SU(N)_1$; we are left with $SU(N)_1$ with $2N$ hypermultiplets, a well-known finite theory with duality [1, 2, 3]. Similarly, using larger values of k, we may construct finite $\mathcal{N}=2$ theories with $[SU(N)]^{k-1}$ groups, which have a row of $k-1$ boxes of finite size and boxes on either end of infinite extent in the x_6 direction. This is shown in fig. 8.

More generally, in any elliptic model whose symmetries do not forbid it, we may take the limit where the torus becomes a cylinder while a set of rows or of columns of boxes retains finite area. For example, we may take the model of fig. 3 in the limit that the x_6 extent of the boxes $(2,1)$ and $(2,2)$ goes to infinity (fig. 9).

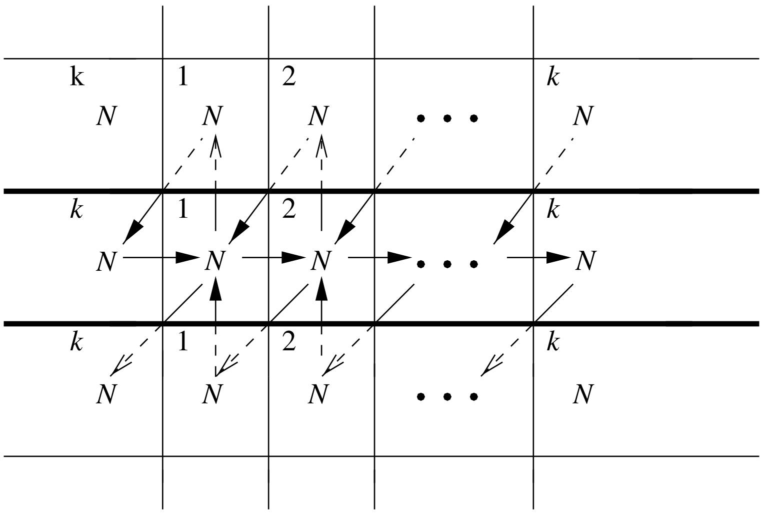

Figure 8: Construction of the $\mathcal{N}=2$ cylindrical model with gauge group $[SU(N)]^{k-1}$. The cylinder is shown as a lattice modulo a unit cell, whose boundary is indicated with a heavy outline.

The boxes in the left column remain interacting, while the right column and its image to the left of the left column become non-interacting flavor groups of the central column. The resulting $\mathcal{N} = 1$ $[SU(N)]^2$ model is finite. There is a bifundamental and biantifundamental of the $SU(N)\times SU(N)$ group and each group factor also has $2N$ flavors from the adjacent flavor groups. Note that all the fields which are neutral under the gauge group decouple from the theory. This model has a flat direction, corresponding to the removal of an NS$'$ brane from the grid, along which it flows to the $\mathcal{N} = 2$ $SU(N)$ with $2N$ hypermultiplets. In a similar way we may obtain finite cylindrical $[SU(N)]^{(k-1)k'}$ theories from elliptic $[SU(N)]^{kk'}$ models.

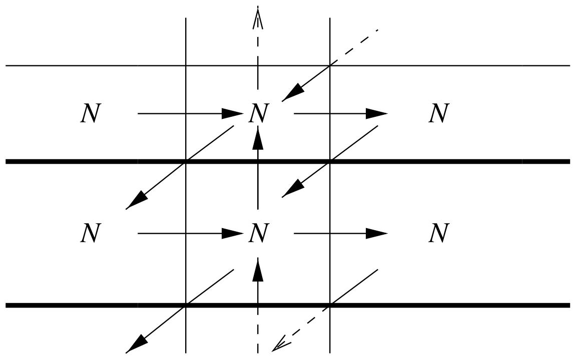

Figure 9: Construction of the $\mathcal{N}=1$ cylindrical model with gauge group $SU(N)\times SU(N)$.

These cylindrical models obviously have fewer marginal couplings than those of the elliptic models of which they are a limit. In particular the Kähler parameter ρ of the torus does not exist for the cylinder.

3.5. Non-finite models with marginal operators

These models are interesting (and to our knowledge have not appeared before) because they contain (1) asymptotically free subgroups, (2) infrared free subgroups, and (3) a completely renormalizable superpotential. The brane picture is less well understood in this case; the branes bend in complicated ways which as yet we do not know how to characterize properly. However, the geometrical simplicity of the brane construction is still useful for building these field theories.

The simplest such model is the torus with 2×2 boxes (with identifications under shifts of two boxes vertically and two boxes horizontally) in which the number of D5 branes is $n_{1,1} = n_{2,2} = N$, $n_{1,2} = N + k$, $n_{2,1} = N - k$. The matter content is as in eq. (2.4). The $SU(N-k)$ group factor has a positive one-loop beta function, the $SU(N+k)$ group factor has a negative one-loop beta function, and so the theory is not finite and indeed is non-renormalizable. However, in the limit that the gauge couplings for all groups except $SU(N+k)$ are zero, the $SU(N+k)$ theory has a conformal fixed point at low energy [10]. Its matter fields have non-trivial anomalous dimensions. By turning on the gauge couplings of the other factors, we can deform this fixed point continously [5]. This fact cannot be seen as yet in the brane picture; our understanding of the bending of the branes is still incomplete. Nonetheless, the field theory story is

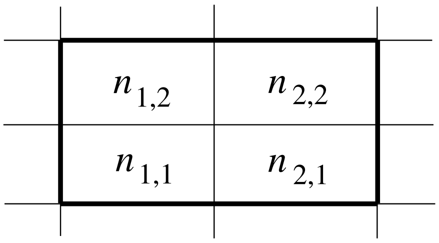

Figure 10: Construction of a non-finite model with a marginal operator; here $n_{1,1} + n_{2,2} = n_{1,2} + n_{2,1}$.

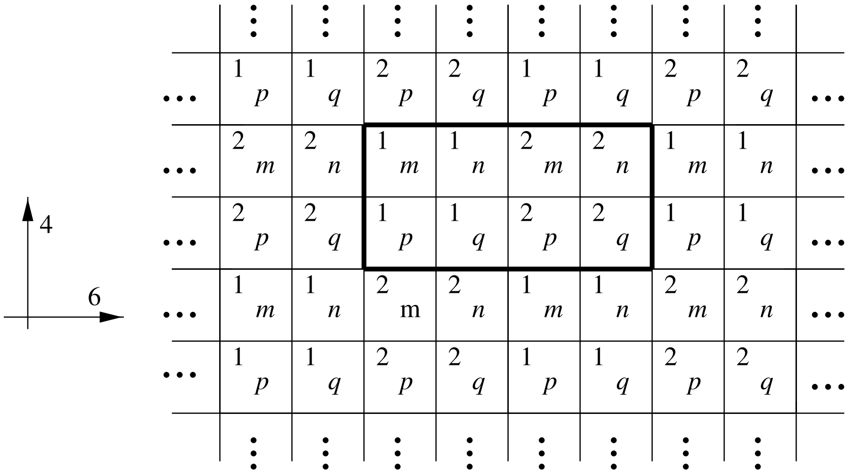

Figure 11: A twisted non-finite model with at least one marginal operator. The theory is $SU(m)_1 \times SU(n)_1 \times SU(p)_1 \times SU(q)_1 \times SU(m)_2 \times SU(n)_2 \times SU(p)_2 \times SU(q)_2$; each box corresponds to one of these factors, as indicated by the indices in the upper left corners.

clear. In fact, it could be studied in detail using the fact that it has a large N limit; for N large and k small, this marginal line begins in the perturbative regime. The theory has two marginal couplings, one from the overall scale of the torus and one from the lower-left/upper-right diagonal line (see Appendix B.)

More generally we can consider a 2×2 model with arbitrary $n_{i,j}$ satisfying $n_{1,1} + n_{2,2} = n_{1,2} + n_{2,1}$. This model is vectorlike and can have a one-dimensional manifold of fixed points. The dynamics may not always permit this, however. For example, if $n_{1,1} = n_{2,1} = 9$ and $n_{2,1} = n_{2,2} = 2$, then the $SU(9)$ groups have 13 flavors and the $SU(2)$ groups have 20 flavors. If we set all but the $n_{1,1}$ coupling to zero, the $SU(9)$ theory with 12 flavors is in the free magnetic phase, and therefore does not reach a fixed point. Indeed a fixed point of this type would be inconsistent with unitarity [10]. Thus one must use this construction with some care; the semiclassical geometry alone is not sufficient.

A similar issue prevents the existence of non-finite $\mathcal{N} = 2$ elliptic models. For example, consider the torus with two boxes containing N_1 and N_2 D5 branes. If $N_1 > N_2$, then the first factor is asympotically free while the second is infrared free. If we turn off the coupling of the second, the first has no fixed point in the infrared corresponding to an unbroken $SU(N_1)$ gauge theory. (There may be Argyres-Douglas fixed points [36] on the moduli space, but these are not governed by naive application of the formulas (1.1)-(1.2); see [37].) Thus there are no solutions to the requirement that all beta functions vanish.

We may also construct chiral models of this type by taking k or k' greater than two. For a twisted example, consider the case $k = 4$, $k' = 2$ where the torus is identified as a grid modulo shifts of four boxes to the right and of two boxes diagonally to the upper right, as in fig. 11. As long as the two 2×2 blocks have the same gauge groups (with eq. (3.1) satisfied), the model will be well defined and have a marginal operator.

Note also that it has a large N limit; if m, n, p, q are of order N while the differences between them are order 1, then the manifold of fixed points of this theory will be visible in perturbation theory.

Many of these theories have flat directions (classically, Fayet-Iliopolous terms) along which they flow to theories with fewer boxes. To remove a single NS or NS$'$ brane we must have the same number of D5 branes on either side of each portion of the NS or NS$'$ brane. In these models, where the numbers of D5 branes may vary from box to box, this condition is generally not satisfied. However, in special cases various branes may be removed; for example, if $n_{i,j} = n_{i+1,j}$, then the brane between the i and $i+1$ columns is not bent and may be lifted off the D5 branes. But there are other flat directions, involving the removal of pairs of NS or NS$'$ branes, which certainly arise. For example, if $n_{i,j} = n_{i+1,j} + p = n_{i-1,j} + p$ for all j, then one may remove the i^{th} and $(i-1)^{th}$ NS branes along with p D5 branes (which wrap the torus in the x_4 direction but are finite in x_6), leaving behind boxes with $n_{i-1,j}$ D5 branes. This corresponds to breaking the theory to a $k-1 \times k'$ box $\mathcal{N}=1$ model times a pure $\mathcal{N}=2$ $SU(p)$ gauge theory.

Finally, let us note that in special cases one may construct interesting non-finite cylindrical $\mathcal{N}=1$ models in analogy to the cylindrical models of the previous section.

The construction of similar $\mathcal{N}=1$ theories which are not finite but have at least one marginal coupling is straightforward. We have not worked out the general counting rules for the number of such couplings, but in any given case these can be worked out using the field theory rules (1.1)-(1.2). Much remains to be understood in the brane picture, including the full story of the bending of the branes and the counting of marginal couplings.

3.6. Still more non-finite models

From certain models — for example, those with $k=2$ — additional non-renormalizable models with quartic superpotentials and marginal couplings, such as in eq. (1.6), can be derived. For $k=2$, the fields $H_{1,j}$ and $H_{2,j}$ are complex conjugates; consequently, a mass term can be written for them, although we do not know how to represent this using branes. In field theory, integrating out these massive fields leads to a gauge theory with the quartic superpotential

$$W = \frac{1}{M}\sum_{j,J}\left[h^+_{1,j}V_{2,j}D_{2,j+1} - h^-_{1,j-1}V_{1,j-1}D_{2,j}\right]\left[h^+_{2,J}V_{1,J}D_{1,J+1} - h^-_{i,J}V_{2,J-1}D_{1,J}\right], \quad (3.7)$$

which, by construction [5], may have dimensionless couplings in its infrared conformal field theory. It is natural to expect $SL(2,\mathbb{Z})$ and its generalizations to act on these couplings, as in certain previously known cases [5, 9]. Note that this construction applies independent of whether the original model is finite.

4. Duality

We consider some examples of the last section with, in turn, their duality properties.

4.1. The $\mathcal{N} = 4$ theory

The simplest case is the $\mathcal{N} = 4$ $SU(N)$ theory, which is realized in our framework as N D5 branes wrapping the T^2 in the 46 directions. As mentioned above, we can remove the NS and NS$'$ branes in this case without changing the theory.

Recall that the gauge coupling constant is given by $1/g_{YM}^2 = (R_4 R_6)/(g_s l_s^2)$, and the theta angle is given by the integral of the RR 2-form over the T^2. Both parameters are arranged in the Kähler parameter ρ of the torus. In this toroidal compactification of the type IIB string theory there is a natural $SL(2,\mathbb{Z})$ action on ρ [38] that leaves the physics invariant, and which corresponds to Olive-Montonen duality in the $\mathcal{N} = 4$ gauge theory. It is interesting to compare this construction to other realizations of the $\mathcal{N} = 4$ theory via branes. After T duality along 4 and 6, the D5 branes become D3 branes sitting at a point in the dual T^2. The geometric factors in $1/g_{YM}^2$ are absorbed in the redefinition of the string coupling; one then has $1/g_{YM}^2 = 1/g_s'$. The integral in 46 of the RR 2-form transforms into the RR scalar. The complex gauge coupling in the gauge theory on the world-volume of the D3 branes is thus given by the type IIB complex coupling constant theory. Thus the T duality maps the Kähler parameter ρ of the initial torus to the string coupling constant in the T dual theory. The $SL(2,\mathbb{Z})$ duality of ten dimensional type IIB string theory [39, 40] accounts for the Olive-Montonen duality on the D3 brane world volume [41, 42].

Another construction of this theory, which is more convenient for our purposes, is obtained by going to an M-theory description. We can perform a T duality along one circle, say x^4; let $x^{4'}$ be the coordinate of the dualized circle. The gauge theory is now realized on the world-volume of N type IIA D4 branes wrapped on the x^6 circle, as in [4]. We then treat type IIA on the T^2 parametrized by $x^{4'}, x^6$ as M theory on a T^3; the D4 branes become M5 branes wrapped on a two torus parametrized by x^6, x^{10}. Denoting by $R_{4'}, R_6$ and R_{10} the lengths of circles in M-theory, the standard relations following from M-theory/type IIB duality [43, 44] are (up to numerical factors)

$$R_4 = \frac{l_{11}^3}{R_{4'} R_{10}}\,; \qquad g_s = \frac{R_{10}}{R_{4'}}\,; \qquad l_s^2 = \frac{l_{11}^3}{R_{10}}\,. \tag{4.1}$$

The gauge coupling in the type IIB construction is now given by $1/g_{YM}^2 = R_6/R_{10}$. The theta angle in the type IIB construction B_{46} becomes, under the T duality, the component along 6 of the IIA RR vector field, A_6, which corresponds to the $6,10$ component of the M-theory metric. These two parameters combine to define the complex structure of the M-theory two-torus along $6,10$. Thus, the transformation to an M-theory description maps the Kähler parameter of the IIB torus to the complex structure of an M-theory torus. As in [4], the geometric $SL(2,\mathbb{Z})$ action on this last accounts for Olive-Montonen duality in the gauge theory.

4.2. $\mathcal{N} = 2$ models

Let us now consider the $\mathcal{N} = 2$ models which are realized by introducing k NS branes (or k' NS$'$ branes) in the type IIB construction, as in fig. 5. The torus along 46 is divided in k boxes, each giving a $SU(N)$ gauge factor. As already stated, the individual gauge couplings and theta parameters are encoded in the positions of the NS branes and the Wilson lines of their world-volume $U(1)$ gauge fields along x^4. As mentioned earlier, these models are the same as the elliptic models of [4], where their duality properties were discussed. We review the construction of the duality group that acts on the parameter space.

The space of parameters is easily visualized by going to the M-theory description as we did just a moment ago. T dualizing along 4 and lifting the IIA configuration to M-theory, the IIB D5 branes become M5 branes wrapping a T^2 parametrized by x^6, x^{10} (denoted as E in what follows), and the NS branes become M5 branes point-like in these coordinates and spanning $4'$ and 5. The Kähler parameter of the type IIB torus becomes the complex structure parameter of E, and the positions and Wilson lines of the NS branes become the positions of the corresponding k M5 branes in 6,10. The space of parameters is the moduli space of genus one smooth Riemann surfaces E with k unordered marked points [4]. It is denoted by $\mathcal{M}_{1,k}$. A point in $\mathcal{M}_{1,k}$ defines a gauge theory with overall coupling constant given by the complex structure on the Riemann surface (equivalently, a parameter τ in the fundamental domain of $SL(2,\mathbb{Z})$), and individual gauge couplings defined by the positions of the k points on the Riemann surface. The fact that the points are unordered reflects the fact that the NS branes are indistinguishable.

Following [4], the duality group is associated to closed loops in this space (changes in the parameters of the theory that leave the physics invariant) with the equivalence relation defined by smooth deformation (homotopy). So the duality group is $\pi_1(\mathcal{M}_{1,k})$ [4]. The simplest way of analyzing what type of duality transformations we are describing is to consider the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}_{1,k}$, and determine the actions we have to quotient $\tilde{\mathcal{M}}$ by to obtain $\mathcal{M}_{1,k}$. These actions constitute the duality group: $\mathcal{M}_{1,k} = \tilde{\mathcal{M}}/\pi_1(\mathcal{M}_{1,k})$.

The space $\tilde{\mathcal{M}}$ can be thought of as $\mathbb{C}^+ \times (\mathbb{R}^2)^k$, where $\mathbb{C}^+$ denotes the classical space of the overall coupling constant, namely the upper half complex plane with no $SL(2,\mathbb{Z})$ identifications. The i^{th} factor $\mathbb{R}^2$ corresponds to the positions of the i^{th} M5 brane on a complex plane. A non-minimal set of generators of $\pi_1(\mathcal{M}_{1,k})$ is given by

- $SL(2,\mathbb{Z})$ transformations on $\mathbb{C}^+$.
- Shifts in the i^{th} $\mathbb{R}^2$ by whole periods of the torus. In the type IIB construction this corresponds to shifting the x^6 positions of the i^{th} NS brane by a multiple of R_6, or, similarly, shifting its Wilson line along 4 by whole periods.
- Permutations of the $\mathbb{R}^2$ factors. This amounts to interchanging the NS branes in the type IIB picture (as in fig. 12.)

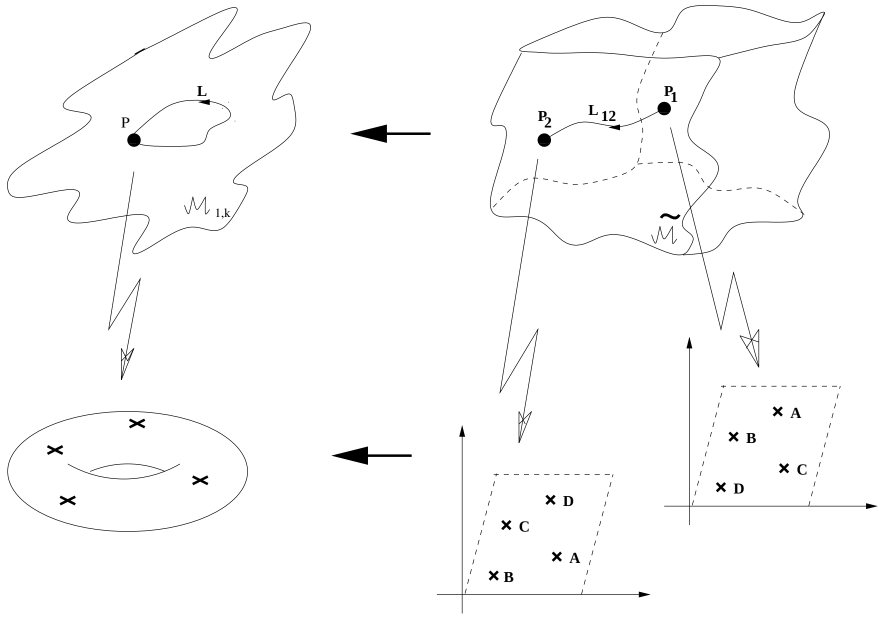

Figure 12: An example of a duality operation: Each point in the universal cover $\tilde{\mathcal{M}}$ represents the choice of coordinates for the M5 branes labelled A, B, C and D. The points P_1 and P_2 differ only in that the branes are permuted. As the M5 branes are *physically* indistinguishable, the two points P_1 and P_2 are a single point P in the space of *physical* parameters $\mathcal{M}_{1,k}$. A curve L_{12} connecting P_1 and P_2 in $\tilde{\mathcal{M}}$ is thus mapped to a homotopically non-trivial closed loop L in $\mathcal{M}_{1,k}$.

Thus the group contains in general not only the overall $SL(2,\mathbb{Z})$ but also additional elements. When the NS branes are removed, leaving the theory $\mathcal{N}=4$ invariant, the overall $SL(2,\mathbb{Z})$ duality group of the k-box model becomes the $SL(2,\mathbb{Z})$ duality group of the $\mathcal{N}=4$ theory.

At this point, it is useful to analyze another M-theory description of the same type IIB $\mathcal{N}=2$ brane configurations. This is obtained by T duality along 6, and transforms the D5-branes into D4-branes wrapping the x^4 circle, and the NS branes into Kaluza-Klein (KK) monopoles, described by a multicentered Taub-NUT space. Notice that the k centers in the Taub-NUT space are coincident, since their x^7, x^8 and x^9 coordinates all vanish. The x^6 positions of the IIB NS branes are encoded in the non-vanishing integrals of the IIA NS-NS 2-form on the k 2-cycles Σ_i of the KK monopole. Going to M-theory by growing a compact dimension $\overline{10}$, the D4-branes become M5-branes wrapping a two-torus along $4, \overline{10}$ (denoted $\overline{E}$), of complex structure parameter ρ (the Kähler parameter of the IIB torus). The multi Taub-NUT metric has translational invariance in $\overline{10}$, and in this sense the KK monopoles wrap $\overline{E}$ as well. The parameters of the IIB NS branes are encoded in the integrals of the M-theory 3-form C_3 over the k $\mathbb{P}_1$'s in the Taub-NUT space times each of the two independent circles in $\overline{E}$,

$$\int_{\Sigma_i \times S^1_{(4)}} C_3\,; \qquad \int_{\Sigma_i \times S^1_{(\overline{10})}} C_3\,. \tag{4.2}$$

These may be called, in a broad sense, Wilson lines along $4, \overline{10}$ of the gauge fields $\int_{\Sigma_i} C_3$. The space of parameters is the moduli space of smooth Riemann surfaces $\overline{E}$ with a choice of k indistinguishable pairs of these Wilson lines. By comparison with the description of the same theory in the previous paragraph, we learn that a choice of a pair of Wilson lines on $\overline{E}$ corresponds to a choice of a point in E. Equivalently, a pair of Wilson lines on $\overline{E}$ lives (takes values) on a two-torus E with the same complex structure. The parameter space is then composed of choices of complex structure and k indistinguishable points in E, again $\mathcal{M}_{1,k}$, and the duality group is its fundamental group. Notice that the compact dimension $6'$ does not play any role in the determination of the duality group.

4.3. $\mathcal{N} = 1$ models

Now we can turn to the analysis of the $\mathcal{N} = 1$ theories, obtained by introducing k NS branes and k' NS$'$ branes. There is a difficulty in the determination of the duality group in this case. In the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ theories, the brane construction displayed very explicitly the marginal operators which were shown to exist using field theory techniques. Thus the space of parameters had a simple geometrical representation, which we uncovered by going to an M theory description. In the generic $\mathcal{N} = 1$ case, however, the field theory analysis shows the existence of additional marginal operators (those associated with diagonal lines of boxes) for which we have not found an interpretation in terms of parameters in the brane configuration. These parameters may be associated to the fields living at the intersection of the NS and NS$'$ branes (as discussed in Appendix A), even though the precise connection is not clear enough to allow for, *e.g.*, matching the counting of these parameters in the field theory and the brane construction. Another possibility is that these parameters may not be realized in the brane configuration.

We leave the precise characterization of the complete parameter space and duality group of these gauge field theories for future work, and, in the following, restrict ourselves to discussing the subspace of parameters which is manifest in the brane construction, and to determining the subset of duality transformations acting on it.

We start with the IIB brane configuration with both NS and NS$'$ branes. Performing a T duality along 4, the D5 branes become M5 branes wrapping the two torus E in $6, 10$, the NS branes become M5 branes pointlike in E, and the NS$'$ branes transform into KK monopoles wrapping E. The compact dimension $4'$ will not play any role. The space of parameters corresponds to choices of a complex structure for E, of k unordered marked points on E (positions of NS branes) and k' unordered marked points on $\overline{E}$ (choice of Wilson lines around E.)[6] Recalling that E and $\overline{E}$ have the same complex structure, and that their Kähler classes are irrelevant, there is no obstruction to identifying them for the purposes of computing the duality group. The space of parameters is then the moduli space of a smooth Riemann surface with k unordered marked points of a certain

[6] In the $\mathcal{N} = 2$ section we showed that Wilson lines around $\overline{E}$ correspond to points in E; an analogous argument shows that Wilson lines around E correspond to points in $\overline{E}$.

kind, and k' unordered marked points of another. We denote this space by $\mathcal{M}_{1,k,k'}$; the associated duality group is $\pi_1(\mathcal{M}_{1,k,k'})$. The full duality group of the theory will contain this as a subgroup.

From our experience with $\mathcal{N}=2$ duality we see that the transformations described include the overall $SL(2,\mathbb{Z})$, shifts of x^4 positions and Wilson lines of NS$'$ branes by whole periods, shifts of x^6 positions and Wilson lines of NS branes by whole periods, and permutations of NS fivebranes of the same kind.

Notice the symmetry between the two kinds of points, corresponding to the equivalence of the M-theory descriptions obtained by T dualizing along 4 or 6 in the initial IIB configuration. Notice also that whenever any kind of NS branes is absent one recovers the duality group for the $\mathcal{N}=2$ theory.

The description of the space of parameters presented above requires further discussion when the fundamental translations defining the type IIB torus are not simple horizontal and vertical shifts, as in the model of fig. 7. As argued above, there exist relations in this case among the couplings of the different gauge factors, and the space of parameters is smaller than what the number of group factors would suggest.

Without loss of generality we can consider a $k \times k'$ box model in which the identification of the vertical sides of the unit cell is trivial (defined by a horizontal translation), but the identification of horizontal sides is accompanied by a horizontal shift of p boxes. Let r denote the greatest common divisor of p and k. The theory can be considered as originating from a $k \times kk'/r$ box model with trivial identifications, modded out by a freely acting $Z_{k/r}$ translation in the torus. This is schematically depicted in fig. 13.

Let us consider the space of parameters of a $k \times kk'/r$ box model, without any $Z_{k/r}$ identification. In the M-theory description we have the torus E with k points on it. The coordinates of these points correspond, when expressed in variables of the type IIB configurations, to pairs (x^6, W_4) of x^6 positions and Wilson lines along 4 of the NS branes. We also have the torus $\overline{E}$, with kk'/r points on it. Their positions correspond, in type IIB language, to the pairs (x^4, W_6) of x^4 positions and Wilson lines along 6 of the NS$'$ branes. Notice how the complex structure parameters τ of the tori are equal because Wilson lines around a circle of radius R live on circles of radius $1/R$.

The $\mathbb{Z}_{k/r}$ symmetry implies that the k points on E actually are grouped as k/r copies of a set of r independent points. Similarly, the kk'/r points on $\overline{E}$ group in k/r copies of a set of k' independent points. The $\mathbb{Z}_{k/r}$ action in the IIB configuration amounts to a simultaneous shift in x^4 (by a fraction r/k of the total period) and x^6 (by a fraction p/k of the total period), so it acts accordingly as a simultaneous shift in E *and* $\overline{E}$, with the correspondent change of copies of marked points in the tori.

The quotient can be taken by choosing a representative for each $\mathbb{Z}_{k/r}$ orbit. This simply means we can use $\mathbb{Z}_{k/r}$ to restrict the range of the x^4 positions of NS$'$ branes to a circle of radius a fraction r/k of the total one, and to only one copy of the points in $\overline{E}$. So we are considering a torus $\overline{E}'$ of complex structure parameter $(r/k)\tau$ (in the type IIB construction, this corresponds to the area of the $k \times k'$ box model being k/r times smaller than the area of the $k \times kk'/r$ box model), and k' points on it. Notice that the range of x^6 positions of points in E cannot be restricted because we have already used all of the $\mathbb{Z}_{k/r}$ symmetry.

Since the radius of x^4 has been reduced, the Wilson lines around 4 live in a circle k/r times larger, so we have to consider a new torus E', instead of the 'old' E. The torus E' also has complex structure parameter $(r/k)\tau$. In this torus there are r independent marked points, and for each we have k/r copies at symmetric positions.

So, the parameters in the theory are: the complex structure for the tori E', $\overline{E}'$, the choice of the k' points in $\overline{E}'$, and the choice of r points in E' (since the addition of the copies implies no choice at all). As before, since the complex structure of the tori is the same, we can identify them as long as we organize the points in two kinds. The space of parameters is then analogous to those analyzed above, $\mathcal{M}_{1,r,k'}$, and the duality group is its fundamental group. In the example in sect. 3, we had $k = 3$, $k' = 1$, $p = 1$, so only the overall $SL(2,\mathbb{Z})$ remains. This agrees with the duality group proposed in [14, 15] for this model.

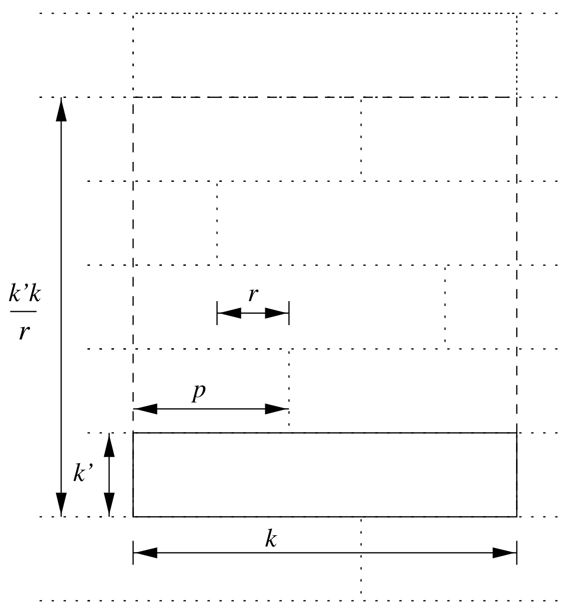

Figure 13: Construction of the $k \times kk'/r$ box model from the $k \times k'$ box model with non-trivial identifications. The unit cell of the $k \times k'$ box model is the solid rectangle. By adjoining a sufficient number of shifted rectangles (in dotted lines) one can define a $k \times kk'/r$ cell (in dashed lines) whose sides are identified in the trivial way. The $k \times k'$ box model is the quotient of the $k \times kk'/r$ box model by a finite group $\mathbb{Z}_{k/r}$ of translations in the torus.

Finally, let us comment on the duality properties of the cylindrical models. These can be obtained by starting with an appropriate elliptic model whose duality group is known, and taking a certain limit in the space of parameters. The duality group of the degenerated model is the subgroup of the original duality group that commutes with this limit.[7]

As a simple example, let us consider the degeneration of an elliptic $\mathcal{N} = 2$ $SU(N)^k$ model to a cylindrical $SU(N)^{k-1}$, introduced above. The space of parameters of the initial model is $\mathcal{M}_{1,k}$. It is easy to see that the limit merely involves the degeneration of the M-theory torus E to a cylinder; the k points remain at finite distance and their parameters remain finite in the limit. The cylinder can be treated as a genus zero Riemann surface with two ordered marked points. Thus, the space of parameters is the moduli space of genus zero Riemann surfaces with $k+2$ marked points, k of which are unordered, and two of which are ordered [4]. The space is denoted $\mathcal{M}_{1,k+2;2}$, and the duality group is its fundamental group.

[7]This statement is familiar from several examples in string duality.

The $\mathcal{N} = 1$ examples can be studied analogously. The simplest degeneration of a $k \times k'$ box elliptic model corresponds to sending to infinity the areas of the boxes in one row or in one column, while keeping the rest of the boxes of finite area, and keeping the NS and NS$'$ branes at fixed positions and with fixed Wilson lines. In the M-theory version, where the structure of the space of parameters is most clear, this corresponds to an (identical) degeneration in the complex structures of the tori E and $\overline{E}$. The space of parameters is the moduli space of a Riemann surface of genus zero with 2 ordered marked points, k unordered marked point of one kind, and k' unordered marked points of another. The duality group of the cylindrical model is the fundamental group of this space.

In conclusion, the brane configurations discussed here provide a very simple construction of a large family of finite $\mathcal{N} = 1$ models and a geometric interpretation of (at least part of) their duality groups. More investigation into the coupling constants is needed, as discussed in Appendix A. Our understanding of the duality groups of the non-finite models is less certain, though it seems likely that the overall $SL(2,\mathbb{Z})$ at least is present.

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A. More on coupling constants

Let us briefly comment on some issues that arise in the definition of the gauge coupling and theta angles of the gauge factors in terms of brane variables. Starting with the simplest case of $\mathcal{N} = 4$ theories, we have seen that the gauge coupling is given by

$$\tau \;=\; \int_{46} B_{RR} + i\frac{R_4 R_6}{g_s l_s^2}\,. \tag{A.1}$$

There are several ways of showing this. An interesting one is to 'measure' τ by computing the action $S = -2\pi i\tau$ for an instanton of the gauge theory. In the brane construction, the instanton corresponds to an euclidean D string wrapping the T^2. The contribution to the real part of the action comes from the area the D string is stretching across, while the imaginary part arises from its coupling to the RR 2-form.

Let us turn to the $\mathcal{N} = 2$ theories. In this case there are several gauge factors, corresponding to the different regions Q_i of the two-torus bounded by the $(i-1)^{th}$ and the i^{th} NS branes. It is easy to see that a naive extension of the previous formula (i.e. τ as the integral of the area and the two-form on the relevant region) cannot be the complete answer. By performing a gauge transformation on the 2-form, $B \to B + d\lambda$, τ would change by a boundary term $\oint_{\partial Q_i} \lambda$. The correct expression can be obtained by computing the action of the euclidean D string stretched across Q_i. It is actually simpler to argue in the IIB S-dual version of the configuration, where a fundamental string stretches between two D5 branes.[8] The only difference with the previous case is a contribution from the world-sheet boundary

$$\oint_{\partial Q_i} A_m \partial_\sigma X^m \,, \tag{A.2}$$

where X^m are the coordinates parametrizing the D5 brane, A_m is a 1-form living on it, and σ parametrizes the world-sheet boundary (in our case ∂Q_i). Since the boundary has two disconnected components l_i, l_{i+1} with different gauge fields A_i, A_{i+1}, the complete expression for the theta angle is

$$\theta_i = \int_{Q_i} B + \oint_b (A_i - A_{i-1}) \,, \tag{A.3}$$

where b denotes the x^4 circle in the torus. This is invariant under the gauge transformation of B provided that the gauge fields transform appropriately under it, $A \to A + \lambda$. Thus in the $\mathcal{N} = 2$ theories, the individual gauge couplings and theta angles are controlled by the positions of the NS branes, and the Wilson lines of their world-volume gauge fields around the compact dimension x^6.

Let us finally turn to the $\mathcal{N} = 1$ theories. Each gauge factor arises from a rectangle R_{ij} bounded in x^6 by the $(i-1)^{th}$ and i^{th} NS branes and in x^4 by the $(j-1)^{th}$ and j^{th} NS$'$ branes. These branes carry gauge fields A_{i-1}, A_i and A'_{j-1}, A'_j. The theta angle has contributions from the rectangle and its boundary

$$\theta_{ij} = \int_{R_{ij}} B + \int_{a_i} (A'_{j-1} - A'_j) + \int_{b_j} (A_i - A_{i-1}) \,, \tag{A.4}$$

where a_i and b_j denote the sides of the rectangle. This is the simplified expression we introduced in sect. 2. It is gauge invariant under transformation of B (with the one-forms compensating the boundary term). However, it is not invariant under the gauge transformations of the one-forms, since the range of line integral for each gauge field is an interval with two points as boundary. The natural solution is that there exist fields $\phi_{i,j}$ living at the intersection of the NS and NS$'$, and which transform like axions under the gauge transformations of the one-forms on the corresponding NS and NS$'$ branes. The complete expression for θ_{ij} then includes an additional contribution $\phi_{i,j} - \phi_{i,j-1} - \phi_{i-1,j} + \phi_{i-1,j-1}$.

[8]When dealing with configurations where the branes bend, this bending depends on the value of the string coupling, and the effect of the S-duality on the brane configuration would be complicated. In our case, however, the branes are undergo no bending, and one can perform the duality without changing the geometry of the configuration.

There are natural candidates for these states, as strings stretching between the i^{th} NS and j^{th} NS$'$ branes. Since there are two directions in which the NS and NS$'$ branes can separate (8 and 9), the states form a hypermultiplet under the $\mathcal{N} = 2$ supersymmetry unbroken by the NS fivebranes at the four dimensional intersection. The presence of the D5 branes breaks supersymmetry further, but only decomposes this $\mathcal{N} = 2$ multiplet in $\mathcal{N} = 1$ multiplets, without projecting out any fields. This is so because the D5 branes do not forbid any brane motion for the NS fivebranes.

In the absence of the D5 branes, many interesting properties of these fields – for instance, the coupling of these fields to the six dimensional fields living on the world-volume of the NS fivebranes – could be analyzed using open string perturbation theory in the S-dual version of the construction. In particular, it should be possible to study how gauge invariance in (A.4) is restored by the contribution from the fields at the D5-D5$'$ intersections, as we have proposed. The contributions of these fields to the gauge coupling constant could also be computed as the action of a fundamental string stretched across the rectangle. However, it is not completely clear whether in this last calculation the effects of the NS branes (S-dual of the original D5 branes) can be ignored.

This type of computation may also help in understanding how the extra marginal operators of the field theory may be represented in the brane picture. We hope that future developments along these lines will allow for a description of the complete parameter space and duality group in $\mathcal{N} = 1$ theories.

B. Proof of marginality

In this section we prove that a model with a grid laid out by NS and NS$'$ branes, and with $n_{i,j}$ D5 branes stretched across the box in the (i, j) position, can have at least one marginal operator. We also prove the existence of additional marginal couplings when $n_{i,j} = N$ for all i, j.

As discussed in sect. 2, each box has a gauge group with a gauge coupling $g_{i,j}$, while each corner at which four boxes intersect has two Yukawa couplings $h^+_{i,j}$, $h^-_{i,j}$. In previous sections we rescaled the holomorphic version of these Yukawa couplings to unity by redefining the fields. Here it is useful to rescale them back into the superpotential so that the techniques of sect. 1 may be used without modification. We define them by rewriting eq. (2.1)

$$W = \sum_{i,j} h^+_{i,j} H_{i,j} V_{i+1,j} D_{i+1,j+1} - \sum_{i,j} h^-_{i,j} H_{i,j+1} V_{i,j} D_{i+1,j+1} \,. \tag{B.1}$$

We will show in this section that

$$\sum_{i,j} A(g_{i,j}) \propto \sum_{i,j} n_{i+1,j} A(h^+_{i,j}) + n_{i,j+1} A(h^-_{i,j}) \,, \tag{B.2}$$

where $A(g), A(h)$ are defined in eqs. (1.1)-(1.2). As discussed in sect. 1, this linear relation allows the theory to have a marginal operator.

Since all of the Yukawa couplings are dimensionless, this relation can only be true if the constant terms in on the left hand side of (B.2) cancel. This sum is

$$\sum_{i,j}(b_0)_{i,j} = \frac{1}{2}\left(6n_{i,j} - n_{i,j+1} - n_{i+1,j} - n_{i-1,j-1} - n_{i,j-1} - n_{i-1,j} - n_{i+1,j+1}\right) , \qquad \text{(B.3)}$$

which clearly vanishes.

It remains to show that the terms proportional to the anomalous dimensions satisfy (B.2). Begin with

$$\begin{aligned}\sum_{i,j} A(g_{i,j}) = \sum_{i,j} \frac{1}{2}\Big[&n_{i,j+1}\gamma(V_{i,j}) + n_{i+1,j}\gamma(H_{i,j}) + n_{i-1,j-1}\gamma(D_{i,j}) \\ &+n_{i,j-1}\gamma(V_{i,j-1}) + n_{i-1,j}\gamma(H_{i-1,j}) + n_{i+1,j+1}\gamma(D_{i+1,j+1})\Big] , \qquad \text{(B.4)}\end{aligned}$$

where $\gamma(\phi)$ is the anomalous mass dimension of the corresponding field (see sect. 1 for definitions.) This can be resummed to read

$$\begin{aligned}\sum_{i,j} A(g_{i,j}) &= \frac{1}{2}\sum_{i,j} \gamma(H_{i,j})\left[n_{i,j} + n_{i+1,j}\right] \\ &\quad +\gamma(V_{i,j})\left[n_{i,j} + n_{i,j+1}\right] + \gamma(D_{i+1,j+1})\left[n_{i,j} + n_{i+1,j+1}\right] \\ &= \frac{1}{2}\sum_{i,j} n_{i+1,j}\left[\gamma(V_{i,j}) + \gamma(H_{i+1,j}) + \gamma(D_{i+1,j+1})\right] \\ &\quad +n_{i,j+1}\left[\gamma(H_{i,j}) + \gamma(V_{i,j+1}) + \gamma(D_{i+1,j+1})\right] \\ &\quad +\gamma(D_{i+1,j+1})\left[n_{i,j} + n_{i+1,j+1} - n_{i+1,j} - n_{i,j+1}\right] , \qquad \text{(B.5)}\end{aligned}$$

Recognizing the first line of the final expression as $n_{i+1,j}A(h^+_{i,j})$ and the second as $n_{i,j+1}A(h^-_{i,j})$, we find that eq. (B.2) is satisfied if eq. (3.1) holds for all (i,j). As noted in sect. 3, gauge anomalies are automatically cancelled, and bending of each NS and NS$'$ brane is minimized.

Now consider the case when $n_{i,j} = N$. Additional marginal couplings follow from the results

$$\begin{aligned}\textstyle\sum_{i=1}^{k} A(g_{i,j}) &= N \textstyle\sum_{i=1}^{k} A(h^+_{i,j}) + A(h^-_{i,j}) \\ \textstyle\sum_{j=1}^{k'} A(g_{i,j}) &= N \textstyle\sum_{j=1}^{k'} A(h^+_{i,j}) + A(h^-_{i,j}) \qquad \text{(B.6)} \\ \textstyle\sum_{p=1}^{P} A(g_{i+p,j+p}) &= N \textstyle\sum_{p=1}^{P} A(h^+_{i+p,j+p}) + A(h^-_{i+p,j+p}) ,\end{aligned}$$

for all i, j, where P is the smallest number with k and k' as a common factor. Since $i = i + P \bmod k$ and $j = j + P \bmod k'$, the boxes $\{i + p, j + p\}$ form a continuous diagonal loop aroung the torus. Each linearly independent relation gives a separate marginal operator; thus we have one marginal coupling for each row, each column, and each diagonal from lower left to upper right. Note the sum of the first line over j and the sum of the second line over i both equal eq. (B.2); thus the rows and columns give $k-1$ and $k'-1$ independent relations respectively. Similarly, the number of independent relations stemming from the third line is one less than kk'/P.

The first relation follows from the following:

$$\sum_{i=1}^{k} A(g_{i,j}) = \frac{1}{2}N\sum_{i=1}^{k}\Big[\gamma(V_{i,j})+\gamma(H_{i,j})+\gamma(D_{i,j})+\gamma(V_{i,j-1})+\gamma(H_{i-1,j})+\gamma(D_{1,j+1})\Big]$$
$$= \frac{1}{2}N\sum_{i=1}^{k}\Bigg(\Big[\gamma(V_{i+1,j})+\gamma(H_{i,j})+\gamma(D_{i+1,j+1})\Big]$$
$$+\Big[\gamma(V_{i,j-1})+\gamma(H_{i-1,j})+\gamma(D_{i,j})\Big]\Bigg)$$
$$= N\sum_{i=1}^{k}\Big(A(h^{+}_{i,j})+A(h^{-}_{i,j})\Big)\ . \tag{B.7}$$

Analogous proofs establish the other relations.

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